Q1

Given the identity

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

prove the following identities:

(i) $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$
(ii) $\cos 2\theta \equiv 1 - 2\sin^2\theta$
(iii) $\cos 2\theta \equiv 2\cos^2\theta - 1$

[4]

$\sin^2\theta + \cos^2\theta \equiv 1$

$\Rightarrow \cos^2\theta \equiv 1 - \sin^2\theta$

$\Rightarrow \sin^2\theta \equiv 1 - \cos^2\theta$

(i) Let $A = B = \theta$, then

$\cos(A+B) = \cos(\theta+\theta) = \cos 2\theta \equiv \cos\theta\cos\theta - \sin\theta\sin\theta$
$= \cos^2\theta - \sin^2\theta$

Therefore $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$

(ii) $\cos^2\theta - \sin^2\theta \equiv (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$

So $\cos 2\theta \equiv 1 - 2\sin^2\theta$

(iii) $\cos^2\theta - \sin^2\theta \equiv \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1$

So $\cos 2\theta \equiv 2\cos^2\theta - 1$

Q2

(i) Prove the identity

$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$$

(ii) Show by counter-example that

$$\cos 3\theta \not\equiv 3\cos\theta - 4\cos^3\theta$$

[5]

$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$ [Compound angle formula]

$\sin 2A \equiv 2\sin A\cos A$ [Double angle formula]

$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$ [Double angle formula]

$\sin^2\theta + \cos^2\theta \equiv 1 \Rightarrow \cos^2\theta \equiv 1 - \sin^2\theta$

(i) $\sin 3\theta = \sin(2\theta+\theta)$, so

$\sin 3\theta \equiv \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$
$\equiv (2\sin\theta\cos\theta)\cos\theta + (1-2\sin^2\theta)\sin\theta$
$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$
$\equiv 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$
$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$
$= 3\sin\theta - 4\sin^3\theta$

Therefore

$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$

(ii) Let $\theta = 0$. Then

$\cos 3\theta = \cos(0) = 1$

$3\cos\theta - 4\cos^3\theta = 3\cos(0) - 4\cos^3(0) = 3(1) - 4(1)^3 = -1$

$1 \neq -1$

Therefore

$\cos 3\theta \not\equiv 3\cos\theta - 4\cos^3\theta$

Q3

Show that

$$\cos 4\theta + \cos\frac{\pi}{3} \equiv 8\sin^4\theta - 8\sin^2\theta + \frac{3}{2}$$

[5]

$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$ [Double angle formula]

$\sin 2A \equiv 2\sin A\cos A$ [Double angle formula]

$\sin^2 A + \cos^2 A \equiv 1 \Rightarrow \cos^2 A \equiv 1 - \sin^2 A$

$\cos 4\theta = \cos(2(2\theta))$, so

$\cos 4\theta \equiv 1 - 2\sin^2 2\theta$
$\equiv 1 - 2(2\sin\theta\cos\theta)^2$
$= 1 - 2(4\sin^2\theta\cos^2\theta)$
$= 1 - 8\sin^2\theta\cos^2\theta$
$\equiv 1 - 8\sin^2\theta(1-\sin^2\theta)$
$= 1 - 8\sin^2\theta + 8\sin^4\theta$

Also $\cos\frac{\pi}{3} = \frac{1}{2}$, so

$\cos 4\theta + \cos\frac{\pi}{3} \equiv (1 - 8\sin^2\theta + 8\sin^4\theta) + \frac{1}{2}$
$= 8\sin^4\theta - 8\sin^2\theta + \frac{3}{2}$

Therefore

$\cos 4\theta + \cos\frac{\pi}{3} \equiv 8\sin^4\theta - 8\sin^2\theta + \frac{3}{2}$

Q4

Prove that

$$\cot^2\theta - \tan^2\theta \equiv 4\cot 2\theta \operatorname{cosec} 2\theta$$

[5]

$\cot A \equiv \frac{\cos A}{\sin A}$ $\tan A \equiv \frac{\sin A}{\cos A}$

$\sin^2 A + \cos^2 A \equiv 1$

$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$ [Double angle formula]

$\sin 2A \equiv 2\sin A\cos A$ [Double angle formula]

$\cot A \equiv \frac{\cos A}{\sin A}$ $\operatorname{cosec} A \equiv \frac{1}{\sin A}$

$$\cot^2\theta - \tan^2\theta \equiv \frac{\cos^2\theta}{\sin^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \frac{\cos^4\theta - \sin^4\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)}{\sin^2\theta\cos^2\theta}$$

$$\equiv \frac{\cos^2\theta - \sin^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{4(\cos^2\theta - \sin^2\theta)}{4\sin^2\theta\cos^2\theta}$$

$$= \frac{4(\cos^2\theta - \sin^2\theta)}{(2\sin\theta\cos\theta)^2}$$

$$\equiv \frac{4\cos 2\theta}{\sin^2 2\theta} = 4\left(\frac{\cos 2\theta}{\sin 2\theta}\right)\left(\frac{1}{\sin 2\theta}\right)$$

$$\equiv 4\cot 2\theta \operatorname{cosec} 2\theta$$

Therefore

$$\cot^2\theta - \tan^2\theta \equiv 4\cot 2\theta \operatorname{cosec} 2\theta$$

Q5

Prove the identity

$$\frac{1-\tan^2 x}{\cos 2x} \equiv \sec^2 x \qquad x \neq \frac{2k+1}{4}\pi$$

[5]

$\tan x \equiv \frac{\sin x}{\cos x}$

$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$ [Double angle formula]

$\sec x \equiv \frac{1}{\cos x}$

$$\frac{1-\tan^2 x}{\cos 2x} \equiv \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{\cos^2 x - \sin^2 x}$$

$$= \frac{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}{\cos^2 x - \sin^2 x}$$

$$= \frac{\cancel{\cos^2 x - \sin^2 x}\ 1}{\cos^2 x\,(\cancel{\cos^2 x - \sin^2 x})}$$

$$= \frac{1}{\cos^2 x}$$

$$= \left(\frac{1}{\cos x}\right)^2$$

$$\equiv \sec^2 x$$

Therefore

$$\frac{1-\tan^2 x}{\cos 2x} \equiv \sec^2 x$$

Q6

Prove the identity

$$\operatorname{cosec} x \equiv \frac{\frac{1}{2}\sec^2\frac{x}{2}}{\tan\frac{x}{2}}$$

[4]

$\operatorname{cosec}\theta \equiv \frac{1}{\sin\theta}$

$\sin 2A \equiv 2\sin A\cos A$ [Double angle formula] $A = \frac{x}{2}$

$\sec\theta \equiv \frac{1}{\cos\theta}$ $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$

$$\boxed{\begin{aligned}\operatorname{cosec} x &\equiv \frac{1}{\sin x}\\ &\equiv \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\\ &= \frac{1/\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}/\cos^2\frac{x}{2}}\\ &= \frac{\frac{1}{\cos^2\frac{x}{2}}}{2\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)}\\ &\equiv \frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2}} = \frac{\frac{1}{2}\sec^2\frac{x}{2}}{\tan\frac{x}{2}}\end{aligned}}$$

Divide top and bottom by $\cos^2\frac{x}{2}$

Therefore

$$\operatorname{cosec} x \equiv \frac{\frac{1}{2}\sec^2\frac{x}{2}}{\tan\frac{x}{2}}$$

Q7

Show that

$$\tan\frac{x}{2} \equiv \frac{1}{\operatorname{cosec} x + \cot x} \qquad x \neq 2k\pi$$

[5]

$\operatorname{cosec} A \equiv \frac{1}{\sin A}$ $\cot A \equiv \frac{\cos A}{\sin A}$

$A = \frac{x}{2}$

$\sin 2A \equiv 2\sin A\cos A$ [Double angle formula]

$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$ [Double angle formula]

$\tan A \equiv \frac{\sin A}{\cos A}$

$$\begin{aligned}\frac{1}{\operatorname{cosec} x + \cot x} &\equiv \frac{1}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}}\\ &= \frac{1}{\frac{1+\cos x}{\sin x}} = \frac{\sin x}{1+\cos x}\\ &= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1+\left(2\cos^2\frac{x}{2}-1\right)}\\ &= \frac{\cancel{2}\sin\frac{x}{2}\cancel{\cos\frac{x}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{x}{2}}\\ &= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\\ &\equiv \tan\frac{x}{2}\end{aligned}$$

Therefore,

$$\tan\frac{x}{2} \equiv \frac{1}{\operatorname{cosec} x + \cot x}$$